Note on Dfferential Invariants of a System of m Points by Projective Transformation.

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As is well known the n^{th} order differential invariants of a geometrical configuration by an r parameter Lie group generated by the r independent infinitesimal transformations

$$U_1 f, \;\; U_2 f, \ldots, \;\; U_r f, \tag{1}$$

are found by integrating the complete system of partial differential equations

$$U_1^{(n)} f = 0, \;\; U_2^{(n)} f = 0, \ldots, \;\; U_r^{(n)} f = 0, \tag{2}$$

where the $U_i^{(n)} f$ are the so-called n^{th} extensions of the original transformations.

In particular if we seek a differential invariant of the second order of a system of m points in the plane by the general projective transformation group of that plane we have to find a solution of the complete system

$$\sum_1^m{}^i \, [X_1'' f]_i = 0, \ldots \sum_1^m{}^i, [X_8'' f]_i = 0, \tag{3}$$

where $[X_j'' f]_i$ represents the result of substituting x_i, y_i for x, y in $X_j'' f$, and $X_1'' f, \ldots, X_8'' f$ are the second extensions of the eight independent infinitesimal transformations which generate the projective group

$$p, \;\; q, \;\; xp, \;\; yp, \;\; xq, \;\; yq, \;\; x^2 p + xyq, \;\; xyp + y^2 q, \tag{4}$$

respectively, p being written for $\dfrac{\partial f}{\partial x}$ and q in place of $\dfrac{\partial f}{\partial y}$.

The second extension of the infinitesimal point transformation

$$X_i f \equiv \xi_i (x, y) p + \eta_i (x, y) q, \tag{5}$$

is

$$X_i'' f \equiv \xi_i p + \eta_i q + \eta_i' q' + \eta_i'' q'', \tag{6}$$

in which

$$\eta' \equiv \frac{d\eta}{dx} - y'\frac{d\xi}{dx}, \quad \eta'' \equiv \frac{d\eta'}{dx} - y''\frac{d\xi}{dx}, \quad y' \equiv \frac{dy}{dx},$$

$$y'' \equiv \frac{d^2y}{dx^2}, \quad q' \equiv \frac{\partial f}{\partial y'}, \quad q'' \equiv \frac{\partial f}{\partial y''}. \tag{7}$$

By means of these expressions the second extensions of the original eight infinitesimal projective transformations are found to be, respectively,

$$\left.\begin{array}{l} p,\, q,\;\; xp - y'q' - 2y''q'',\;\; yp - y'^2q' - 3y'y''q'',\;\; xq + q',\;\; yq + y'q' + y''q'' \\ x^2p + xyq + (y - xy')\,q' - 3xy''q'',\;\; xyp + y^2q + y'\,(y - xy')\,q' - 3xy'y''q''. \end{array}\right\} \tag{8}$$

Then those functions $\phi\,(x_1,\, y_1,\, y'_1,\, y''_1,\, \ldots\,,\, x_m,\, y_m,\, y'_m,\, y''_m)$, which are differential invariants of the second order of a system of m points $(x_1,\, y_1,\, \ldots\,,\, x_m,\, y_m)$, are solutions of the complete system of equations

$$\left.\begin{array}{l} \displaystyle\sum_1^m{}_i \frac{\partial\phi}{\partial x_i} = \sum_1^m{}_i \frac{\partial\phi}{\partial y_i} = \sum_1^m{}_i \left(x_i\frac{\partial\phi}{\partial x_i} - y'_i\frac{\partial\phi}{\partial y'_i} - 2y''_i\frac{\partial\phi}{\partial y''_i}\right) \\ \displaystyle\qquad = \sum_1^m{}_i \left(y_i\frac{\partial\phi}{\partial x_i} - y'^2_i\frac{\partial\phi}{\partial y'_i} - 3y'_iy''_i\frac{\partial\phi}{\partial y''_i}\right) = \sum_1^m{}_i \left(x_i\frac{\partial\phi}{\partial y_i} + \frac{\partial\phi}{\partial y'_i}\right) = 0, \\ \displaystyle\sum_1^m{}_i \left(y_i\frac{\partial\phi}{\partial y_i} + y'_i\frac{\partial\phi}{\partial y'_i} + y''_i\frac{\partial\phi}{\partial y''_i}\right) \\ \displaystyle\qquad = \sum_1^m{}_i \left\{x_i^2\frac{\partial\phi}{\partial x_i} + x_iy_i\frac{\partial\phi}{\partial y_i} + (y_i - x_iy'_i)\frac{\partial\phi}{\partial y'_i} - 3x_iy''_i\frac{\partial\phi}{\partial y''_i}\right\} \\ \displaystyle\qquad = \sum_1^m{}_i \left\{x_iy_i\frac{\partial\phi}{\partial x_i} + y_i^2\frac{\partial\phi}{\partial y_i} + y'_i(y_i - x_iy'_i)\frac{\partial\phi}{\partial y'_i} - 3x_iy'_iy''_i\frac{\partial\phi}{\partial y''_i}\right\} = 0. \end{array}\right\} \tag{9}$$

This simultaneous system has at least $4m - 8$ independent solutions; direct integration yields that m of these solutions are of the form

$$\sum_1^m{}_i \frac{y''_k}{y''_i}\left\{\frac{y_k - y_i - y'_i(x_k - x_i)}{y_k - y_i - y'_k(x_k - x_i)}\right\}^3, \quad k = 1,\, 2, \ldots\,,\, m. \tag{10}$$

In order to interpret these invariants geometrically, take m curves in the plane perfectly arbitrarily except that a curve is to pass through each of the m points of the system which we are studying; let ρ_i be the radius of curvature of the curve through $(x_i,\, y_i)$ at $(x_i,\, y_i)$; take any point $(x_k,\, y_k)$ of the system of points and join it by straight lines to all the other points of the system; let θ_i be the angle at $(x_k,\, y_k)$ between the normal to the curve through $(x_k,\, y_k)$ and the line

joining (x_k, y_k) to (x_i, y_i), and let ϕ_i be the angle between the latter line and the normal to the curve through (x_i, y_i); then the expressions (10) show that the m forms

$$\sum_1^m {}_i \frac{\rho_k \cos^3 \theta_i}{\rho_i \cos^3 \phi_i} \quad i \neq k, \ (k = 1, 2, \ldots, m) \quad (11)$$

are absolute invariants under the general projective group.

If, in particular, the m points lie on a straight line, these invariants reduce to the simpler forms

$$\sum_1^m {}_i \frac{\rho_k \cos^3 \theta_k}{\rho_i \cos^3 \theta_i}, \quad i \neq k, \quad (k = 1, 2, \ldots, m). \quad (12)$$

By means of the theorem of Reiss, stating that $\sum_{i}^{m} \rho_i^{-1} \cos^{-3} \theta_i$ is zero when a straight line cuts a curve of the m^{th} degree, we have that the value of the invariant is minus unity in case the m points lie on a straight line and a curve of the m^{th} degree simultaneously, and conversely.

Let the given system of points lie on a straight line and let the curves be so chosen that they are normal to the straight line at the points of the system. The invariants then become

$$\sum_1^{m-1} {}_i \frac{\rho_k}{\rho_i} \text{ or } \sum_1^m {}_i \frac{1}{\rho_i} = 0. \quad (13)$$

This includes as special cases the theorem that the ratio of the radii of curvature at corresponding points of two parallel curves is unaltered by projective transformation and also the theorem, given as new by Wölffing* but due to H. J. Stephen Smith,† that the ratio of the radii of curvature of two tangent curves at the point of tangency, is unaltered by projective transformation; it is obvious that the latter theorem appears without insisting that the curves be normal to the line.

In view of the fact that the general projective transformation group preserves its form under a transformation from point coordinates to line coordinates (the individual transformations are not invariant, but the group as a whole), corres-

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* Wölffing, "Das Verhältniss der Krümmunsgsradien im Berührungspunkte zweier Curven," Zeitschrift für Mathematik und Physik, Bd. XXXVIII, 1893, pp. 237–249.

† H. J. Stephen Smith, "On the Focal Properties of Homographic Figures," Proc. London Mathematical Society, vol. II, pp. 196–248.

ponding theorems relative to the invariants of a system of m lines can be immediately written down from the above forms.

Consider a system of m points $(x_1, y_1, z_1, \ldots, x_m, y_m, z_m)$ in ordinary space. A differential invariant of this system by an infinitesimal point transformation

$$Vf \equiv \xi(x, y, z)\frac{\partial f}{\partial x} + \eta(x, y, z)\frac{\partial f}{\partial y} + \zeta(x, y, z)\frac{\partial f}{\partial z} \tag{14}$$

is a solution,

$$\psi(x_1, y_1, z_1, p_1, q_1, r_1, s_1, t_1, \ldots; \ldots, x_m, y_m, z_m, p_m, \ldots) \tag{15}$$

of the partial differential equation

$$\sum_1^m {}_i\,[V^{(n)}f]_i = 0,$$

where $V^{(n)}f$ is the n^{th} extension of Vf.

In order to determine the form of this extended transformation we proceed as follows:* Putting

$$p \equiv \frac{\partial z}{\partial x}, \quad q \equiv \frac{\partial z}{\partial y}, \quad r \equiv \frac{\partial^2 z}{\partial x^2}, \quad s \equiv \frac{\partial^2 z}{\partial x \partial y}, \quad t \equiv \frac{\partial^2 z}{\partial y^2}, \ldots, \tag{16}$$

the variation of the identity

$$dz \equiv pdx + qdy \tag{17}$$

gives

$$d\delta z \equiv \delta p \,.\, dx + \delta q \,.\, dy + pd\delta x + qd\delta y. \tag{18}$$

δx, δy, δz are given functions of x, y, z from (14), namely,

$$\delta x = \xi(x, y, z)\delta\varepsilon, \quad \delta y = \eta(x, y, z)\delta\varepsilon, \quad \delta z = \zeta(x, y, z)\delta\varepsilon, \tag{19}$$

where $\delta\varepsilon$ is an arbitrary infinitesimal of the first order. The identity (18) is to exist for all values of dx and dy, hence it breaks up into two equations for determining δp and δq, the variations of p and q by the infinitesimal transformation Vf. Calling these $\pi\delta\varepsilon$ and $\varkappa\delta\varepsilon$ respectively, the first extension of the transformation (14) is

$$V'f \equiv \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z} + \pi\frac{\partial f}{\partial p} + \varkappa\frac{\partial f}{\partial q}, \tag{20}$$

in which

$$\left.\begin{aligned} \pi &= \zeta x + p\zeta_z - p(\xi_x + p\xi_z) - q(\eta_x + p\eta_z), \\ \varkappa &= \zeta_y + q\zeta_z - p(\xi_y + q\xi_y) - q(\eta_y + q\eta_z). \end{aligned}\right\} \tag{21}$$

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* See Lie, "Vorlesungen über continuierliche Gruppen," bearbeitet und herausgegeben von Scheffers, Leipzig, 1893, pp. 709, 710.

Similarly the variation of the identities

$$dp \equiv r dx + s dy, \quad dq \equiv s dx + t dy \tag{22}$$

yields equations for the variations δr, δs, δt by the transformation Vf. The solution of these equations gives for the second extension of Vf,

$$V''f \equiv V'f + \rho \frac{\partial f}{\partial r} + \sigma \frac{\partial f}{\partial s} + \tau \frac{\partial f}{\partial t}, \tag{23}$$

in which

$$\left.\begin{aligned}
\rho &= \pi_x + p\pi_z + r\pi_p + s\pi_q - r(\xi_x + p\xi_z) - s(\eta_x + p\eta_z), \\
\sigma &= \pi_y + q\pi_z + s\pi_p + t\pi_q - r(\xi_y + q\xi_z) - s(\eta_y + q\eta_z) \\
&= \varkappa_x + p\varkappa_z + r\varkappa_p + s\varkappa_q - s(\xi_x + p\xi_z) - t(\eta_x + p\eta_z), \\
\tau &= \varkappa_y + q\varkappa_z + s\varkappa_p + t\varkappa_q - s(\xi_y + q\xi_z) - t(\eta_y + q\eta_z).
\end{aligned}\right\} \tag{24}$$

The higher extensions are computed in the same way. We have now the implements in hand by which to prosecute the study of the problems in space corresponding to those already discussed in the plane.

The formulæ (21) and (24) give the following forms to the second extensions of the fifteen independent infinitesimal transformations of the general projective group of ordinary space:

$$\left.\begin{aligned}
&\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial z}, \quad \frac{x\partial f}{\partial x} - p\frac{\partial f}{\partial p} - 2r\frac{\partial f}{\partial r} - s\frac{\partial f}{\partial s}, \\
&\frac{x\partial f}{\partial y} - q\frac{\partial f}{\partial p} - 2s\frac{\partial f}{\partial r} - t\frac{\partial f}{\partial s}, \quad \frac{x\partial f}{\partial z} + \frac{\partial f}{\partial p}, \\
&y\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial q} - r\frac{\partial f}{\partial s} - 2s\frac{\partial f}{\partial t}, \\
&y\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial q} - s\frac{\partial f}{\partial s} - 2t\frac{\partial f}{\partial t}, \quad y\frac{\partial f}{\partial z} + \frac{\partial f}{\partial q}, \\
&z\frac{\partial f}{\partial x} - p^2\frac{\partial f}{\partial p} - pq\frac{\partial f}{\partial q} - 3pr\frac{\partial f}{\partial r} - (2ps + qr)\frac{\partial f}{\partial s} - (2qs + pt)\frac{\partial f}{\partial t}, \\
&z\frac{\partial f}{\partial y} - pq\frac{\partial f}{\partial p} - q^2\frac{\partial f}{\partial q} - (2ps + qr)\frac{\partial f}{\partial r} - (2qs + pt)\frac{\partial f}{\partial s} - 3qt\frac{\partial f}{\partial t}, \\
&z\frac{\partial f}{\partial z} + p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} + r\frac{\partial f}{\partial r} + s\frac{\partial f}{\partial s} + t\frac{\partial f}{\partial t}, \\
&x^2\frac{\partial f}{\partial x} + xy\frac{\partial f}{\partial y} + zx\frac{\partial f}{\partial z} + (z - px - qy)\frac{\partial f}{\partial p} \\
&\qquad\qquad - (3rx + 2sy)\frac{\partial f}{\partial r} - (2sx + ty)\frac{\partial f}{\partial s} - tx\frac{\partial f}{\partial t},
\end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned}
&xy\frac{\partial f}{\partial x}+y^2\frac{\partial f}{\partial y}+yz\frac{\partial f}{\partial z}+(z-px-qy)\frac{\partial f}{\partial q}-rx\frac{\partial f}{\partial r}\\
&\qquad\qquad\qquad-(2sy+rx)\frac{\partial f}{\partial s}-(3ty+2sx)\frac{\partial f}{\partial t},\\
&zx\frac{\partial f}{\partial x}+yz\frac{\partial f}{\partial y}+z^2\frac{\partial f}{\partial z}+p(z-px-qy)\frac{\partial f}{\partial p}+q(z-px-qy)\frac{\partial f}{\partial q}\\
&\quad-\{3prx+(qr+2ps)y\}\frac{\partial f}{\partial r}-\{(qr+2ps)x+(pt+2qs)y\}\frac{\partial f}{\partial s}\\
&\quad-\{3qty+(pt+2qs)x\}\frac{\partial f}{\partial t}.
\end{aligned}\right\}\text{—Continued.}\ (25)$$

From these forms we write directly the simultaneous system of partial differential equations

$$\begin{aligned}
&\sum_1^m{}_i\frac{\partial\phi}{\partial x_i}=\sum_1^m{}_i\frac{\partial\phi}{\partial y_i}=\sum_1^m{}_i\frac{\partial\phi}{\partial z_i}\\
&\qquad=\sum_1^m{}_i\left(x_i\frac{\partial\phi}{\partial x_i}-p_i\frac{\partial\phi}{\partial p_i}-2r_i\frac{\partial\phi}{\partial r_i}-s_i\frac{\partial\phi}{\partial s_i}\right)=\ldots\ldots=0,
\end{aligned}\qquad(26)$$

to be satisfied by the second order differential invariants of the system of m points. The complete system is one of fifteen equations in $8m$ variables; hence there are at least $8m-15$ invariant functions. By integrating the system by successive substitutions, m of these invariants are found to have the following form:

$$\sum_1^m{}_i\frac{s_k^2-r_kt_k}{s_i^2-r_it_i}\left\{\frac{z_i-z_k-p_i(x_i-x_k)-q_i(y_i-y_k)}{z_i-z_k-p_k(x_i-x_k)-q_i(y_i-y_k)}\right\}^4,\qquad k=1,2,\ldots\ldots,m.\qquad(27)$$

These invariants are susceptible of a geometrical interpretation quite analogous to that given in the plane. Take m surfaces in space perfectly arbitrarily chosen except that a surface is to pass through each of the m points of the system; let R_i and R_i' be the principal radii of curvature of the surface through (x_i, y_i, z_i) at (x_i, y_i, z_i); take any point (x_k, y_k, z_k) of the system of points and join it by straight lines to all the other points of the system; let θ_i be the angle at (x_k, y_k, z_k) between the normal to the surface through (x_k, y_k, z_k) and the line joining (x_k, y_k, z_k) to (x_i, y_i, z_i); and let ϕ_i be the angle between the latter line and the normal to the surface through (x_i, y_i, z_i); then the expressions (27) show

that the m forms

$$\sum_{1}^{m}{}_{i}\frac{R_k R_k' \cos^4 \theta_i}{R_i R_i' \cos^4 \phi_i}, \qquad (28)$$
$$i \neq k, \quad k = 1, 2, \ldots, m$$

are absolute invariants by the general projective group.

When the m points lie on a straight line these m invariants reduce to the simpler forms

$$\sum_{1}^{m}{}_{i}\frac{R_k R_k' \cos^4 \theta_k}{R_i R_i' \cos^4 \theta_i}, \qquad (29)$$
$$i \neq k, \quad k = 1, 2, \ldots, m$$

If the m points lie on a straight line and a surface of the m^{th} degree simultaneously, we have

$$\sum_{1}^{m}{}_{i}\frac{1}{R_i R_i' \cos^4 \theta_i} = 0$$

as a generalization for ordinary space of the theorem of Reiss for the plane.

If the m points are collinear and the surfaces be so chosen that they are normal to the line the invariants become

$$\sum_{1}^{m}{}_{i}\frac{R_k R_k'}{R_i R_i'}, \qquad (30)$$
$$i \neq k, \quad k = 1, 2, \ldots, m$$

This last result shows that to generalize the theorem of Smith relative to tangent curves and the theorem given relative to parallel curves it is only necessary to substitute "surface" for "curve" and "measure of curvature" for "radius of curvature." As a matter of fact there is nothing in the reasoning to forbid allowing the m points to fall together, and if we proceed to this limit from the case of collinearity and m surfaces normal to this line, the form (30) obtains in the limit and expresses an invariant relation by projective transformation in the curvatures of m surfaces tangent at a common point.

PRINCETON, N. J., 21 *April*, 1898.